MTH 166

Lecture-3

Equations Reducible to Exact Form-II

Topic:

Equations Reducible to Exact Form

Learning Outcomes:

1. How to find an I.F. using different methods (Next Two Methods).

2. How to convert a Non-exact equation into an Exact equation using I.F. and find its solution.

Methods to find I.F.

3. If equation $Mdx + Ndy = 0$ (1)

can be re-written in the form: $f(xy)ydx + g(xy)xdy = 0$

then $I.F. = \frac{1}{Mx - Ny}$ provided $Mx - Ny \neq 0$

Solve the following differential equations:

Problem 1. $\boldsymbol{(y + xy^2)dx + (x - x^2y)dy = 0}$

Solution: $(y + xy^2)dx + (x - x^2y)dy = 0$ (1)

Compare it with: $Mdx + Ndy = 0$

Here $M = (y + xy^2) \Rightarrow \frac{\partial M}{\partial y} = 1 + 2xy$

and $N = (x - x^2y) \Rightarrow \frac{\partial N}{\partial x} = 1 - 2xy$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

Given equation is: $(y + xy^2)dx + (x - x^2y)dy = 0$ (1)

This equation can be re-written as:

$(1 + xy)ydx + (1 - xy)xdy = 0$

Which is of the form: $f(xy)ydx + g(xy)xdy = 0$

So, $I.F. = \frac{1}{Mx - Ny} = \frac{1}{(y+xy^2)x-(x-x^2y)y} = \frac{1}{2x^2y^2}$

Multiplying equation (1) by I.F., we get:

$[(y + xy^2)dx + (x - x^2y)dy] \times \frac{1}{2x^2y^2} = 0$

$\Rightarrow \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$

Which is exact differential equation.

Solution: $\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$

$\Rightarrow \int_{y=con.} \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx - \int \frac{1}{2y}dy = c$

$\Rightarrow -\frac{1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$

Problem 2. $\boldsymbol{\left(x^2y^3 - y\right)dx + \left(x^3y^2 + x\right)dy = 0}$

Solution: $(x^2y^3 - y)dx + (x^3y^2 + x)dy = 0$ (1)

Compare it with: $Mdx + Ndy = 0$

Here $M = (x^2y^3 - y) \Rightarrow \frac{\partial M}{\partial y} = 3x^2y^2 - 1$

and $N = (x^3y^2 + x) \Rightarrow \frac{\partial N}{\partial x} = 3x^2y^2 + 1$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

This equation can be re-written as:

$(1 + xy)ydx + (1 - xy)xdy = 0$

Which is of the form: $f(xy)ydx + g(xy)xdy = 0$

So, $I.F. = \frac{1}{Mx - Ny} = \frac{1}{(x^2y^3 - y)x - (x^3y^2 + x)y} = -\frac{1}{2xy}$

Multiplying equation (1) by I.F., we get:

$$[(x^2y^3 - y)dx + (x^3y^2 + x)dy] \times \frac{-1}{2xy} = 0$$

$$\Rightarrow \left(-\frac{xy^2}{2} + \frac{1}{2x}\right)dx + \left(-\frac{x^2y}{2} - \frac{1}{2y}\right)dy = 0$$

Which is exact differential equation.

Solution: $\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$

$$\Rightarrow \int_{y=con.} \left(-\frac{xy^2}{2} + \frac{1}{2x}\right)dx - \int \frac{1}{2y}dy = c$$

$$\Rightarrow -\frac{x^2y^2}{4} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

Methods to find I.F.

4. For an equation: $Mdx + Ndy = 0$ (1)

(I) If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, then $I.F. = e^{\int f(x)dx}$

(II) If $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$, then $I.F. = e^{\int f(y)dy}$

Solve the following differential equations:

Problem 1. $\boldsymbol{(xy^3 + y)dx + (2x^2y^2 + 2x + 2y^4)dy = 0}$

Solution: $(xy^3 + y)dx + (2x^2y^2 + 2x + 2y^4)dy = 0$ (1)

Compare it with: $Mdx + Ndy = 0$

Here $M = (xy^3 + y) \Rightarrow \frac{\partial M}{\partial y} = 3xy^2 + 1$

and $N = (2x^2y^2 + 2x + 2y^4) \Rightarrow \frac{\partial N}{\partial x} = 4xy^2 + 2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

Here $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{(4xy^2+2)-(3xy^2+1)}{(xy^3+y)} = \frac{xy^2+1}{xy^3+y} = \frac{(xy^2+1)}{y(xy^2+1)} = \frac{1}{y} = f(y)$

So, $I.F. = e^{\int f(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$

Multiplying equation (1) by I.F., we get:

$$[(xy^3 + y)dx + (2x^2y^2 + 2x + 2y^4)dy] \times y = 0$$

$$\Rightarrow \; (xy^4 + y^2)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0$$

Which is an exact differential equation.

Solution: $\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$

$$\Rightarrow \int_{y=con.} (xy^4 + y^2)dx + \int 2y^5 dy = c$$

$\Rightarrow \frac{x^2y^4}{2} + xy^2 + 2\frac{y^6}{6} = c$ **Answer.**

Problem 2. $\boldsymbol{\left(4xy + 3y^2 - x\right)dx + \left(x^2 + 2xy\right)dy = 0}$

Solution: $(4xy + 3y^2 - x)dx + (x^2 + 2xy)dy = 0$ (1)

Compare it with: $Mdx + Ndy = 0$

Here $M = (4xy + 3y^2 - x) \Rightarrow \frac{\partial M}{\partial y} = 4x + 6y$

and $N = (x^2 + 2xy) \Rightarrow \frac{\partial N}{\partial x} = 2x + 2y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

Here $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(4x+6y)-(2x+2y)}{(x^2+2xy)} = \frac{2x+4y}{(x^2+2xy)} = \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x} = f(x)$

So, $I.F. = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

Multiplying equation (1) by I.F., we get:

$$[(4xy + 3y^2 - x)dx + (x^2 + 2xy)dy] \times x^2 = 0$$

$$\Rightarrow \ (4x^3y + 3x^2y^2 - x^3)dx + (x^4 + 2x^3y)dy = 0$$

Which is an exact differential equation.

Solution: $\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$

$$\Rightarrow \int_{y=con.} (4x^3y + 3x^2y^2 - x^3)dx + \int (0)dy = c$$

$$\Rightarrow \frac{4x^4y}{4} + 3\frac{x^3}{3}y^2 - \frac{x^4}{4} = c$$

$$\Rightarrow 4x^4y + 4x^3y^2 - x^4 = 4c$$

$\Rightarrow 4x^4y + 4x^3y^2 - x^4 = c_1$ **Answer.** $(4c = c_1)$